Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 8 November 2022 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^{n} i^q = 1^q + 2^q + 3^q + \ldots + n^q \text{ where } n, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n, for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. [1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^{n} i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^{n} i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p. [1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^{n} i^2 = a_1 n + a_2 n^2 + a_3 n^3 \text{ where } a_1, a_2, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1, a_2 and a_3. [3]

(iii) Hence, find the values of a_1, a_2 and a_3. [2]

(This question continues on the following page)

(Question 1 continued)

You will now consider a method that can be generalized for all values of q.

Consider the function $f(x) = 1 + x + x^2 + \ldots + x^n$, $n \in \mathbb{Z}^+$.

(c) Show that $x f'(x) = x + 2x^2 + 3x^3 + \ldots + nx^n$. [1]

Let $f_1(x) = x f'(x)$ and consider the following family of functions:

$$f_2(x) = x f_1'(x)$$

$$f_3(x) = x f_2'(x)$$

$$f_4(x) = x f_3'(x)$$

$$\ldots$$

$$f_q(x) = x f_{q-1}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^{n} i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^{n} i^q x^i$, $q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \ldots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \dfrac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

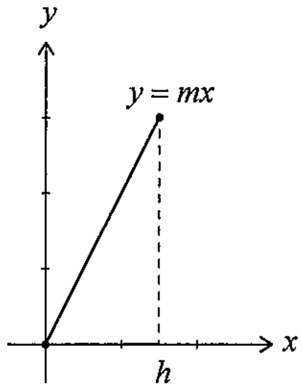
(g) (i) Show that $\lim_{x \to 1} f_1(x)$ is in indeterminate form. [1]

(ii) Hence, by applying l'Hôpital's rule, show that $\lim_{x \to 1} f_1(x) = \dfrac{1}{2} n(n+1)$. [5]

2. [Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \le x \le h$ and m, h are positive constants.



When this line is rotated through $360°$ about the x-axis, a cone is formed with a curved surface area A given by:

$$A = 2\pi\int_0^h y\sqrt{1+m^2}\,\mathrm{d}x .$$

(a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\,\pi$. [2]

(b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \le x \le h$ through $360°$ about the x-axis.

(i) Deduce an expression for the radius of this cone r in terms of h and m. [1]

(ii) Deduce an expression for the slant height l in terms of h and m. [2]

(iii) Hence, by using the above integral, show that $A = \pi r l$. [3]

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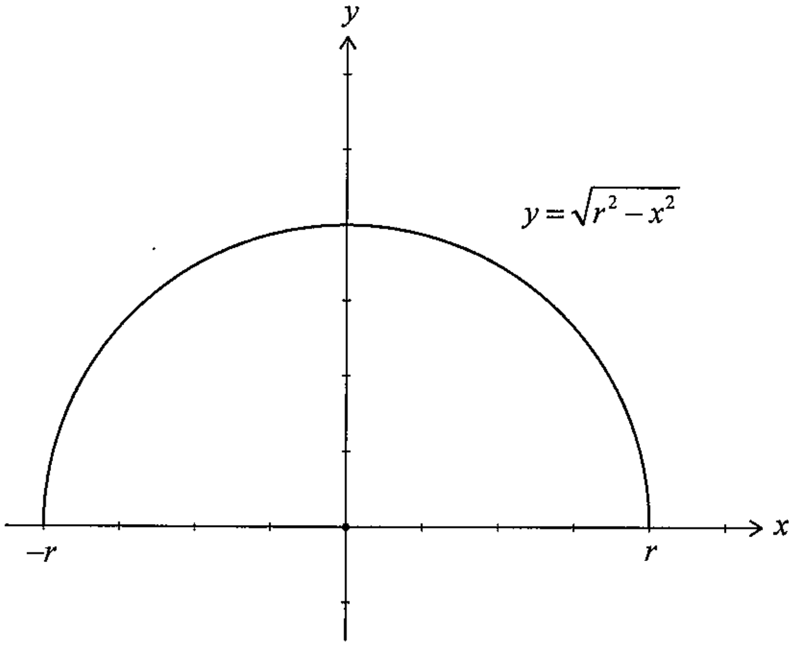
(Question 2 continued)

Consider the semi-circle, with radius r, defined by $y=\sqrt{r^2-x^2}$ where $-r \le x \le r$.



(c) Find an expression for $\frac{dy}{dx}$. [2]

A differentiable curve $y=f(x)$ is defined for $x_1 \le x \le x_2$ and $y \ge 0$. When any such curve is rotated through $360°$ about the x-axis, the surface formed has an area A given by:

$$A = 2\pi \int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx .$$

(d) A sphere is formed by rotating the semi-circle $y=\sqrt{r^2-x^2}$ where $-r \le x \le r$ through $360°$ about the x-axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

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(Question 2 continued)

(e) Let $f(x) = \sqrt{r^2 - x^2}$ where $-r \le x \le r$.

The graph of $y = f(x)$ is transformed to the graph of $y = f(kx)$, $k > 0$. This forms a different curve, called a semi-ellipse.

(i) Describe this geometric transformation. [2]

(ii) Write down the x-intercepts of the graph $y = f(kx)$ in terms of r and k. [1]

(iii) For $y = f(kx)$, find an expression for $\frac{dy}{dx}$ in terms of x, r and k. [2]

(iv) The semi-ellipse $y = f(kx)$ is rotated $360°$ about the x-axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A, of the ellipsoid. Give your answer in the form $2\pi \int_{x_1}^{x_2} \sqrt{p(x)}\, dx$, where $p(x)$ is a polynomial. [4]

(v) Planet Earth can be modelled as an ellipsoid. In this model:

- the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
- the distance from the North Pole to the South Pole is $12\,714\,\text{km}$.
- the diameter of the equator is $12\,756\,\text{km}$.

By choosing suitable values for r and k, find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a \times 10^q$ where $1 \le a < 10$ and $q \in \mathbb{Z}^+$. [4]